3Cセもか76

次の極限値を求めよ。

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\sqrt{1+\frac{1}{n}}+2\sqrt{1+\frac{2}{n}}+3\sqrt{\frac{3}{n}}+\cdots+n\sqrt{1+\frac{n}{n}}\right)$$

〔近畿大〕

$$与式=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k\sqrt{1+\frac{k}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\sqrt{1+\frac{k}{n}}$$

$$=\int_0^1 x\sqrt{1+x}\,dx \cdots ①$$ $\sqrt{1+x}=u$ とおくと

$1+x=u^2$, $dx=2u\,du$, $x=u^2-1$ $x:0\to1$, $u:1\to\sqrt{2}$

これより①は

$$①=\int_1^{\sqrt{2}}(u^2-1)\cdot u\cdot 2u\,du$$

$$=\int_1^{\sqrt{2}}(2u^4-2u^2)\,du$$

$$=\left[\frac{2}{5}u^5-\frac{2}{3}u^3\right]_1^{\sqrt{2}}$$

$$=\left(\frac{8}{5}\sqrt{2}-\frac{4}{3}\sqrt{2}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)$$

$$=\frac{4}{15}\sqrt{2}+\frac{4}{15}$$